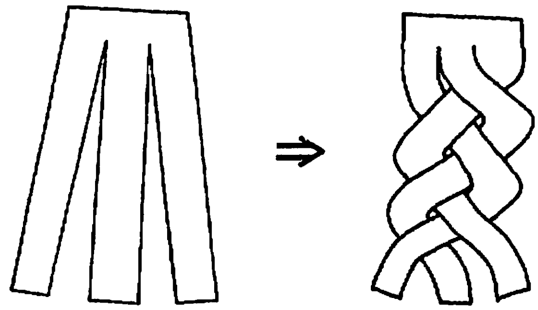
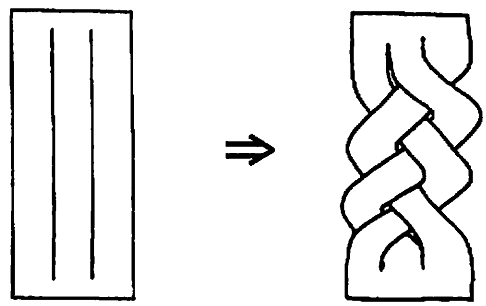
5.3 Bizarre Braids

One usually makes a braid with three strands that are joined at one end but free at the other.



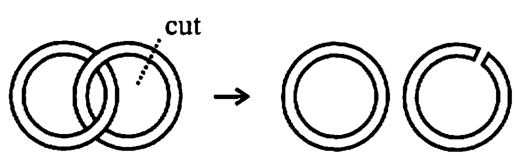
Is it possible to make a braid with no free ends?



Comment. You can use a slit rectangle of paper, but felt is more flexible and easier to manipulate.

5.4 Linked Unlinked Rings

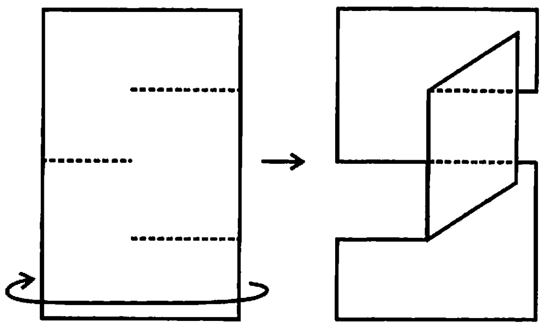
Two linked rings have the property that if you cut either of them, the configuration falls into two separate pieces.



Is it possible to interlink three rings of paper in such a way that if you cut any one of them (once) the configuration is guaranteed to fall into three separate pieces?

5.2 A Mysterious Flap

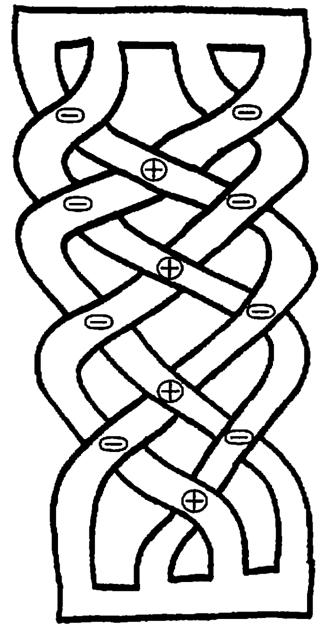
Continuing from the hint, a simple 180° rotation of the bottom half of paper produces the mysterious flap!



Advertise your next public event with signs made this way. I guarantee they will garner plenty of attention and invoke much conversation. Plus, they have the advantage of being conspicuous from great distances along corridors! To make signs like these, photocopy the text and decoration on *two sides* of the paper as shown below before cutting and twisting.

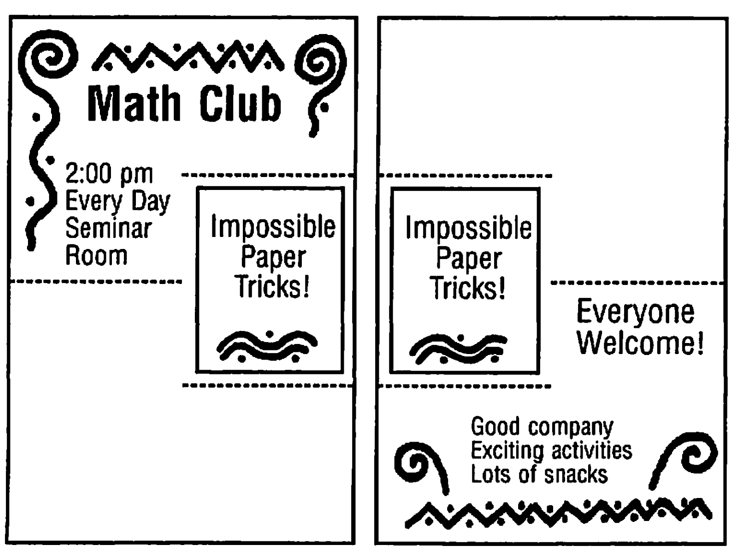
5.3 Bizarre Braids

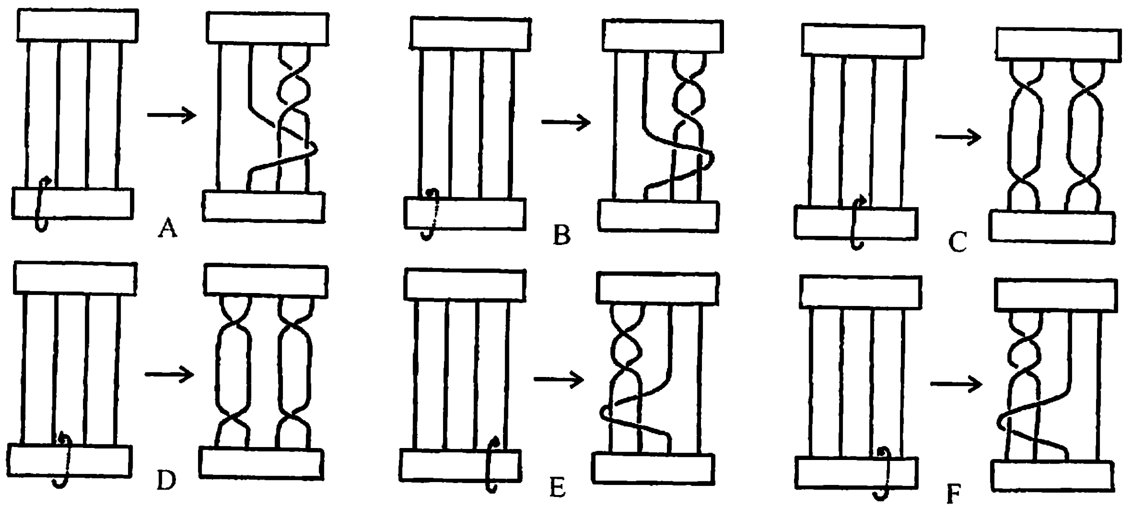
It is not possible. Here's an intuitive argument why not. Let's call the crossover of two adjacent strands *positive* if the left strand crosses over the right, *negative* otherwise. Let the number of positive crossovers minus the number of negative crossovers be alled the *index* of a braid. Thus the four-braid we are trying to create has index −4. To obviate the issue of twists in strands, let's assume we are working with four thin strings attached at both ends to small squares of cardboard . Certainly if it is impossible to braid these four strings, it is impossible to braid four thick strands.



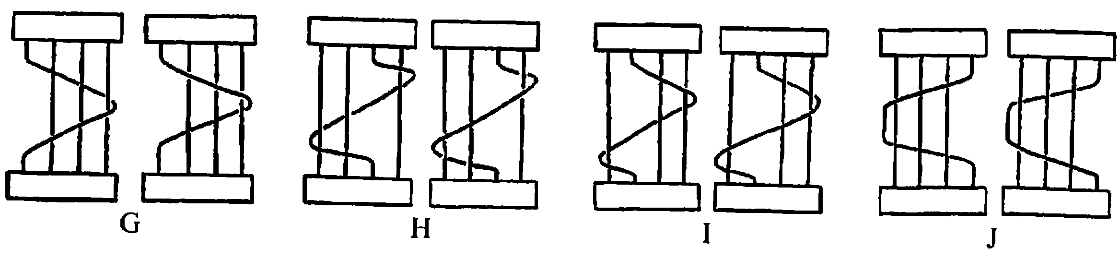
$8\ominus + 4\oplus = -4$

There are a number of basic maneuvers we could perform on this system. We could rotate

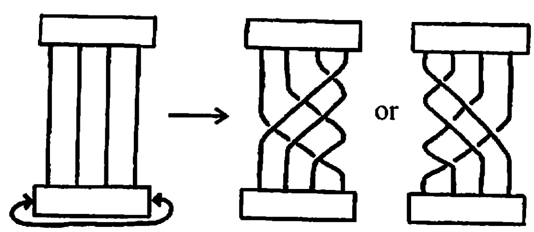




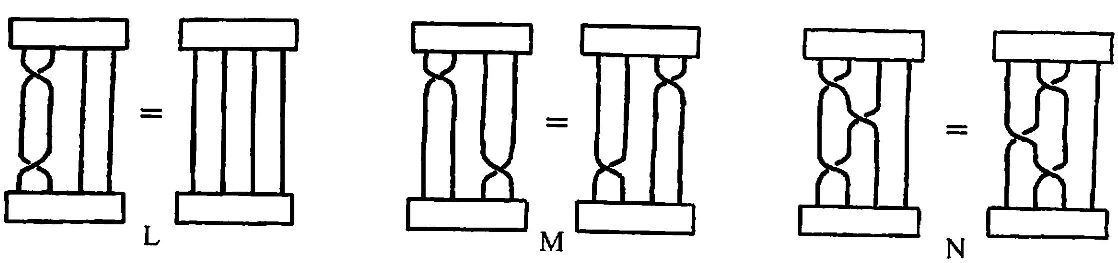
the bottom square of cardboard, back or forth, between two strands, as shown in A–F. We could pick one strand and rotate it, back and forth, around the bottom square of cardboard, as G–J illustrate.



We could rotate the bottom square about a vertical axis, back or forth, as shown in K.



Or we could perform some fundamental maneuvers within a given diagram, as examples L–N suggest.



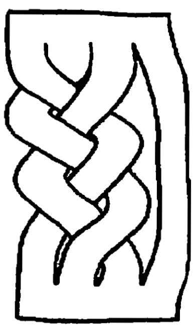
Each maneuver (except for the last three) introduces new crossovers into any given braid diagram and changes its index by +6, −6 or 0. We started with a diagram of four untangled strings, index 0, and hoped to achieve one of index −4. As every maneuver changes a diagram's index by a multiple of six, the task is impossible.

Comment 1. This argument can be made rigorous with the aid of Emil Artin's theory of braids. See [Arti1], [Arti2], and especially [Newm].

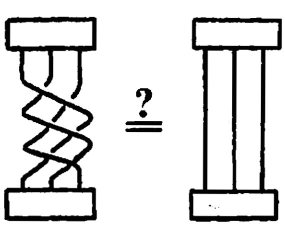
Comment 2. Tie three strings to the back of a chair and braid them (with free ends) in some complicated manner. (A braid is formed as a sequence of twists of two adjacent strands.) Just make sure when you are done that the end of the leftmost string ends up at the leftmost position, the end of the middle string in the middle, and the end of the rightmost string at the right.

Now tape these three ends to a small piece of cardboard making sure to preserve their order. Is it possible to untangle your braid with the three ends now fixed in place? It turns out the answer is always yes! No matter how complicated a braid you make with three free strands, exactly the same feat can be accomplished with the three ends tied together! Try it! (See [Shep].)

Challenge 1. Is it possible to make the following braid with no free ends? (Again, see [Shep].)

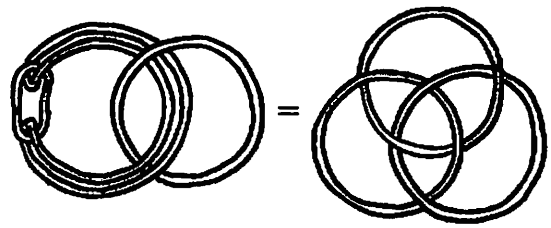


Challenge 2. Three strings are attached to a teacup and to three points about a room. The teacup is given a full turn of 360°. Without moving the teacup, prove it is impossible to maneuver the strings around the cup and untangle them . (See Section 12.1 and [Newm].)

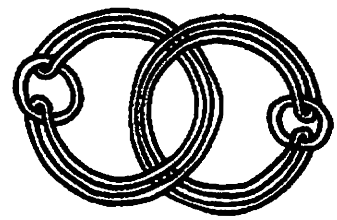


5.4 Linked Unlinked Rings

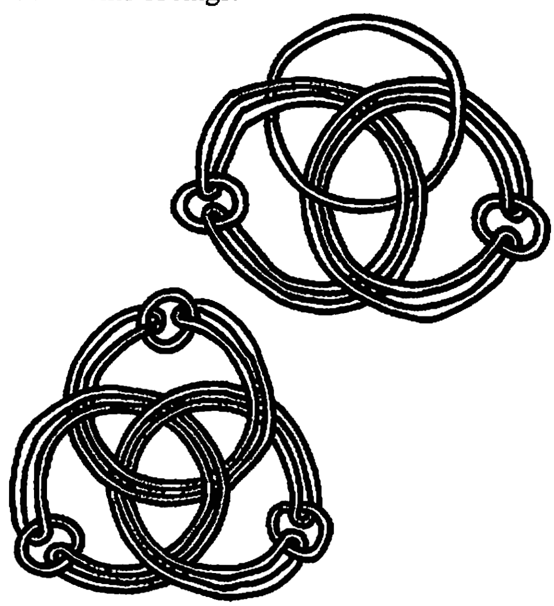
By replacing one ring in a pair of linked rings by a double band of rings we obtain the Borromean rings.



A foursome of linked unlinked rings is obtained from the construction shown.



It is also possible to construct a five-some and a six-some of rings:



Challenge. Can you construct seven or more linked unlinked rings?

Acknowledgments and Further Reading

I first learned of the "mysterious flap" from an advertisement for a social function with the