0.1 Liar's Bingo - KLEIN

Participants examine a handful of paper strips covered with red and black numbers. After identifying any patterns they see, the facilitator performs a magic trick (plays Liar's Bingo), and the hunt begins for the "secret" to the magic trick. This session involves recognizing pattern and searching for underlying structure, number theory, numeration, and potentially binary arithmetic.

Grade Level/Prerequisites: This activity is suitable for participants in grades 6-∞.

Time: This activity is flexible and could fit nicely in one 50-minute session or be expanded to two 50-minute sessions.

Materials: Liar's Bingo cards. These can be downloaded at http://www.mathteacherscircle.org/wp-content/themes/mtc/assets/liars-bingo.pdf, printed in color, and cut into strips. The facilitator should do this in advance because the strips are printed in a convenient order that reveals too much to the participants if they cut up the strips.

Grouping: This activity works best when 4-5 participants can huddle around a clear table where they can lay out the cards, sort them, etc.

Preparation: It's best to practice ahead of time with a fellow teacher who is in on the trick to make sure that you can do the trick relatively quickly and consistently. This trick depends on the facilitator being able to determine the number the participant is lying about every time.

Objectives: Participants will:

- ... discover patterns and thereby look for and make use of the underlying structure of the Liar's Bingo cards;
- ... reason using number bases other than base-10, notably base 2 (binary), and gain further facility with place-value systems.

References/Authorship: This session was shared with me by Mr. Steve Phelps, Madeira High School. He learned it from Chuck Sonenshine (http://www.keynotemath.com/).

LIAR'S BINGO

The human brain is an amazing pattern-processing machine. Many games depend on this and the best music, art, and stories use disruption of patterns to add interest–think of syncopation in music (perhaps a switch to a minor key in an upbeat tune), a bright red ballon held by a little girl in an otherwise black-and-white photograph, or the twist of a plot when we learn that Luke Skywalker has a sister. Sometimes, as in the game of Liar's Bingo, order seems to arise magically from something we first assume to be random or chaotic. In this case, we use the game of Liar's Bingo to engage participants' desire to find patterns, and supercharge that desire by demonstrating a magic trick that captivates attention by disrupting participants' expectations that the Liar's Bingo cards are without pattern.

I have led this activity more than twenty times with audiences ranging from fifth-grade participants to retirees, with backgrounds ranging from not yet ready for fraction arithmetic to PhDs in mathematical research. As my colleague Steve Phelps points out, "Patterns are the great leveler," and Liar's Bingo is suitable for that fifth grader and the PhD in mathematics to work side-by-side discovering the patterns at relatively the same pace. It exemplifies what Jo Boaler calls "low-floor, high-ceiling" problems: everyone can get started (low-floor) and everyone can find challenge (high-ceiling).

Don't be surprised if every time you lead this session, that a participant points out a new pattern in the cards that you have never seen before. Of the 20+ times I've led this, only once have I not been presented with a pattern previously unknown to me. This makes it fun and fresh for the facilitator each time.

(1) *Objective 1: Get participants to engage with the cards and to begin noticing patterns.* Don't mention the "magic trick" at this stage. Launch the activity by asking participants if they would like to learn a new game called "Liar's Bingo." I usually then tell them that I'm going to hand them a handful of strips of paper and that they should lay them out on the table in front of them. There are two prompts that work well for launching this. Either (1) ask them to put the cards at their table in order, or (2) ask them to write down everything that they observe about the cards in front of them, no matter how straight-forward. In the case of (1), participants will inevitably ask what you mean by "in order" and you should instruct them, "I'm not going to say. Whatever 'in order' means to you." In the case of (2), participants often start by looking for something deep, but here *everything* is important so find a way to encourage them to note that, for instance, there are six "cells" of numbers, or that

some of the numbers are red and some black. The objective of this (10-15 minutes) is to get the class to note as many observations or patterns as they can find.

(2) *Objective 2: Collect these patterns for the whole class.* Have groups share out, perhaps noting the patterns on the classroom board or chart paper for all to see. In some cases, a document camera may be helpful as participants have to use cards and pointing to evidence the pattern they see.

(3) *Objective 3: Introduce the game.* I usually ask participants to "keep these patterns/observations in mind" and that "they may be useful and we're going to revisit them." Ask participants if they would like to learn to play Liar's Bingo. In order to play Liar's Bingo, you must first teach them to play Truth-teller's Bingo. Show the class a card that has a mixture of red and black numbers, such as the one below (a document camera may be helpful here if it is available). To play Truth-teller's bingo we read the colors only (not the numbers) from left to right. In the case of the card below, we would say, "Black, Red, Black, Black, Red, Red."

63	3	33	27	21	22

Pick another card and have participants play Truth-teller's Bingo one more time. When you are certain that they understand, congratulate them on being great players of Truth-teller's Bingo and assure them that they are ready to learn Liar's Bingo. The game of Liar's Bingo is just like that of Truth-teller's Bingo except that the player lies about *just one color*. For instance, in the card above, the player might decide ahead of time that they are going to lie about the third cell from the left (the 33) and they would therefore read "Black, Red, Red, Black, Red, Red." Pick another card, pick a cell, and ask them, "if we were to lie about this cell [indicating one of the cells], how would we read this card?"

When you are convinced that they understand, ask someone across the room to play Liar's Bingo to convince you that you understood. Have their table make sure that he/she is playing correctly. After they read the colors, ask that participant's table, "did he/she do it correctly?" If they say yes, then respond with, "so he lied about 27?" (or whatever the corresponding number is). Since you could not see the card and since the participant only read the colors, they will be astounded. Let the room understand what just happened. Sometimes when the participant is reading the colors I will pretend to be busy, not looking at them, by erasing the board with my back to them, adding to the effect. Invite more participants to play against you and guess the number each time. I usually ask them how they think I know. "Do I have ESPN?" (this usually gets a laugh). "Did I memorize *all* of the cards?"

(4) *Objective 4: participants should uncover the trick.* In some instances, the class may need overnight to continue working on this. After 5-8 correct guesses during the class, tell them that they are going to have to use those patterns they figured out (in Objective 1) to understand the trick. Give them some time to talk in their groups and circulate, offering to play with anyone. The following prompts help participants to focus on keys to unlocking the trick:

- What's the smallest number you see? The largest? What's the smallest *digit* you see? The largest?
- What would be a useful sequence of colors to ask me about? (Going for all black (0) or all red (77) here as a starting point). Have participants write these down.
- If you were to lie to me about zero, what would you say to me? About one? Two?
- participants usually develop the notation BRBBRR (using letters to represent the colors) but if they don't, you might encourage its use.

Timing is hard to predict and every group of participants is different. As a few participants indicate that they "get" the trick, encourage their peers to test them by having those participants play and guess the number. Use those participants to help others begin to understand the trick (not to tell them *how to do it* but to *help them discover it*). You may have the feeling that the participants will never get it, but in the many times I have led it, almost all of the participants understand this by the end of 1-1.5 hours. Avoid the urge to say too much – they have far more fun when left to discover it with your nudging than if you tell them how to do it.

(5) *Objective 5 (Optional but important): Connect the understanding of* how to do the trick *with the* patterns identified in Objective 1. There *is* underlying structure that makes the trick work and this structure is coded into the cards. I have had participants who claim to understand the trick go home and make their own set of cards to convince me. I have also given participants something like the card below, with only the first cell completed and asked, "Is this enough? Can you tell me what all the

other cells should be based on just that cell?" (Without referring to the pre-made cards).

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Alternatively, you might give them a card with only the third or other cell shown and ask the same question. A participant who can complete the card above could be challenged to do the same with a card that begins with a black 23 (instead of the red 23 above).

Depending on the list of patterns identified, some important connections should probe:

- why we need two colors. What if we had three colors?
- how many cards constitute a complete set?
- why there are no digits higher than 7 and no numbers higher than 77?
- why we have six cells. (How) could we use this *same* system to represent three digit numbers? Which three digit numbers could we (not) represent in that case?

These connections (to the underlying structure and patterns of the existing cards) and modifications of the existing system (more colors, more cells, etc.) are investigations in their own right that give participants ways to keep thinking about the session. In classroom use, they could be extensions for participants who are working at a faster pace and want further challenge in class. They could be also be used in a computer programming context to challenge participants to code the algorithms necessary to (1) create a complete set of cards and/or (2) play the game. Steve Phelps has programed GeoGebra, for instance, to supply a random Liar's Bingo card each time a button is clicked. Indeed, the session's connections to binary and base-8 representation are evidence of the game's connections to computer engineering–a connection worth making to the participants (a guide to this can be found at http://www.bbc.co.uk/education/guides/zwsbwmn/revision/1.

Teacher Guide & Solutions

Liar's Bingo is, at its heart, about finding joy in discovering patterns and using them to unlock a mystery. The facilitator's role then is to provide just enough nudging to allow participants the time and space to experience the joy in discovering. There is a fine line between *productive struggle* and *frustration leading to resignation*. Facilitators should celebrate insights widely to keep participants motivated and believing that they will figure it out. Understanding some of the patterns that participants may notice and struggles they may encounter will help facilitators to guide participants in this way, to keep things oriented toward fun and productive struggle. The notes below may be helpful to that end and are given in list format for easy reference.

Understanding the trick:

There are several ways to understand the trick. Perhaps the simplest involves dividing each color sequence in half (the first three colors and the last three colors). Understand that the first three colors determine the tens digit of the number being guessed and the last three colors determine the ones (unit) digit of the number being guessed. When a participant gives you the sequence, BRRRBR, the first three colors (BRR) determine the tens digit, and the last three colors (RBR) determine the ones digit.

To determine the tens digit and ones digit, we use binary numbers. Recall that for base-ten numeration, we have names for the first ten whole numbers (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) and then we use place value to represent any other whole number (other number sets aren't discussed here). In base-ten, our place value positions proceed according to powers of ten from right to left: ones, tens, hundreds, etc., or numerically, $10^0 = 1,\ 10^1 = 10,\ 10^2 = 100$, etc. Hence, the number 3140 represents 3 thousands, 1 hundred, four tens, and 0 ones.

Binary numeration is simlar, except that it records the numbers using only two digits (0 and 1) with place-value positions according to powers of two. So instead of ones, tens, hundreds, we have ones, twos, fours, eights, and so on, or $2^0 = 1,\ 2^1 = 2,\ 2^2 = 4,\ 2^3 = 8$, etc. Hence, the number 11010 would represent 1sixteen, 1 eight, 0 fours, 1 two, and 0 ones. We know this in base ten as the number $16 + 8 + 2 = 26$. If we wanted to count in base-2 (binary) up to 26, it would look like: 0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111, 10000, 10001, 10010, 10011, 10100, 10101, 10110, 10111, 11000, 11001, 11010.

In Liar's Bingo, we are using Red to indicate a '1' and Black to indicate a '0.' Hence writing the first seven numbers in binary $(0, 1, 10, 11, 100, 101, 110, 111)$

can be done using these colors: B, R, RB, RR, RBB, RBR, RRB, RRR. It takes only three cells to represent the numbers 0 to 7. This is a key insight for why there are no 8's or 9's on the cards and why the Liar's Bingo cards can represent the set of numbers present on the cards.

So the sequence RBR represents $4+0+1=5$. If these are the first three colors of the six-color sequence, they represent $5\times 10=50$. If they are the last three colors of the six-color sequence, then they represent 5. Hence, the sequence RBR RBR represents the number 55.

Let's understand the sequence BRRRRR by looking at the first three colors and the last three colors separately. The first three colors are BRR indicating 0 fours, 1 two, and 1 one, or $2+1=3$. But since these first three colors represent the tens-digit of the number we are guessing, we know that the "magic" number is thirty-something. The last three colors in the sequence are RRR indicating 1 four, 1 two, and 1 one, or $4+2+1=7$. Because these represent the ones digit of the "magic number", we now know that the number the participant lied about is 37.

So if a participant says BBBBBB we know there are 0 tens and 0 ones (represented by BBB and BBB respectively) so the number lied about is 0. If they give the sequence RRRRRR then they are lying about the number 777 (RRR indicates 7, in this case 7 tens and 7 ones).

Test yourself. If a participant was lying about the number 63, what would they say? Do it now. The answer can be found above in the participant notes in the strip provided there. It's best to practice a lot with a colleague to develop speed and consistency.

Notes by Objective

(1) *Objective 1: Get participants to engage with the cards and to begin noticing patterns.*

- *The Launch Prompt.* It is difficult to know in advance whether the first prompt ("order the cards"), or the second ("what do you notice?") will work best. Having participants order the cards (whatever that might mean to them) is often a more focused task and can work better for participants who might not have had much experience with "what do you notice?" questions before. That said, if in some alternate reality I were asked to lead Liar's Bingo at a conference for poets, I might use the second prompt since the idea of ordering might be more intimidating than that of creative observation. The choice should be motivated by the facilitator's knowledge of the participants and their background. I often start with the first (order these) prompt and then use the second (what do you notice) prompt as I walk around the room from group to group.

Groups will almost always order the cards numerically and in increasing order according to the first cell. Some striking patterns can emerge from this (see below). Note, though, that this is not the only interpretation of order. I have had groups order by getting all cards with a zero and lining them up so that the zeros form a diagonal (zero in the first cell in the first card, in the second cell on the second card, and so on); they then do similarly for cards with a '1' and so on. Other groups have ordered them according to the number of red cells or black cells. In each case, interesting patterns emerge.

- *A sample of patterns:*
  - There are six cells.
  - Some numbers are black and some are red.
  - The smallest digit is 0 and the largest is 7. There are no 8's or 9's on any card.
  - The largest number is 77.
  - Black numbers decrease left-to-right and red numbers increase left-to-right.
  - The first three cells (left-to-right) have equal ones digits and the last three cells have the same tens digit (in some cases an implied zero for the tens digit).

  Each of the patterns the class generates can be *empirically tested* as a conjecture. When a group offers a pattern such as "Black numbers decrease left-to-right," tell the participants that we now have a *conjecture* and we can test that conjecture by looking at the cards for a *counterexample*. Someone may ask whether or not you have handed out "all of the cards," that is, a complete set. Assure them that you have. In such a case, conjectures can be verified by the whole class determining if a counterexample exists.

  Each of the patterns the class generates may be connected later to the structure underlying the trick. This session demonstrates well that *knowing how is not always knowing why.* In this case, participants might know *how* to do the trick, but not understand the underlying structure that explains *why* the trick works and where the patterns above come from.

(2) *Objective 2: Collect these patterns for the whole class.*
Ask each group to contribute one pattern or observation they noticed regarding the cars. Keep polling the groups until you have them all (or until you deem it important to move on). The facilitator, having walked around the room during the observation time should be aware of what the groups noticed and can help motivate them to contribute if necessary. Keep each group engaged by asking them to look at their cards to verify or to find a counter-example each time a group makes an observation. For instance, "Look at your cards. Does anyone have an example where the black numbers do NOT decrease reading left to right?" Use vocabulary like "conjecture" and "counterexample" as they are key to reasoning mathematically.

(3) *Objective 3: Introduce the game.*
If you were careful not to mention "magic trick" before then participants should be astounded when you guess the number. At some point the participant might ask about common features of games that aren't actually present here, like, "how do you win?" or "do you take turns?" I usually tell them simply, "it's not that kind of game." It is important to move past the "magical wonder" and shock they have at your ability to correctly guess the number to focus them on wondering how the trick is done. Push them to ignore the cards after a while and to start to think about the sequence of colors and what makes sense to ask (like all black or all red).
One misconception that arises each time I do this is that participants will look at the all black strip 40-20-10-4-2-1 and wonder where zero is. Why does BBBBBB represent zero when there isn't a zero on the all black strip. Remind them that when we say, "BBBBBB," that we are lying about one of the colors. That is, the person who says, "BBBBBB" has to be looking at a card with one red (and lying about it).
Participants will also want to know if they can lie about the colors of more than one cell at a time. The answer is, "no." We are trying to grow great mathematicians after all, not great liars.

(4) *Objective 4: participants should uncover the trick.* This objective is the hardest to determine how long it takes. Some participants discover the trick more quickly than others. Have these participants' fellow group members play the game with them to be sure that they understand. Then use them to help guide the others toward discovery (this is different from simply teaching the others how to do it). Sometimes this portion of the session takes 20 minutes and sometimes it takes longer. In some

cases, it has taken 30 minutes to have most understand how to do the trick. As mentioned above, celebrate people's observations in the service of continuing their motivation to keep discovering the trick. Emphasize using notes and writing down sequences of colors and the values they represent. This is a good time to make a table of sequences and values (BBBBBB is 0, BBBBBR is 1, and so on).

(5) *Objective 5 (Optional but important): Connect the understanding of* how to do the trick *with the* patterns identified in Objective 1.
In the 1-1.5 hour sessions I have used in the past, we don't always get to work on these, but it is important to emphasize that the patterns identified by the class can be explained by the underlying structures of the cards, including properties of binary numbers and the use of $3+3$ cells to represent the $tens + ones$ digits of the "magic number." Some notes below explaining the patterns above should get you started. It's okay not to know in advance all of the patterns and why they might not work. I frequently am confronted with a pattern someone discovers in this activity that I can't explain at the outset. It's good to mention new patterns when they arise and to admit when you can't explain it. But it is not okay to shirk it off–encourage the group as a whole to consider those moments important open problems that the group should work on. Be sure to follow up in subsequent sessions to see if anyone made progress on those open problems.
Sample patterns from above:

- **There are six cells.** We only need three cells to represent the digits 0-7 but we have two decimal places so we need three cells for the ones and three cells for the tens. If we wanted to expand this to three-digit numbers (using only 0-7 as our digits) then we would need $3+3+3=9$ cells to represent the number. Hence BRRBBRRRB would represent the number 316. It might be fun to encourage students to manufacture a set of 9-celled Liar's Bingo cards. Students often ask about using $4+4=8$ cells (four cells for the tens digit and four cells for the ones digit) so that we could then represent 8s and 9s. What would happen if we did that? Some of the numbers wouldn't have unique representations. For instance, looking only at the four cells representing the ones place, what would RRBR represent? In binary, we know that this would be 1 eight, 1 four, 0 twos, and 1 one, or $8+4+1=13$. Can we have a 13 in the ones place? Consider two cards:

BBBBRRBR and BBBRBBRR

Both of these sequences represent the number 13 even though they are different sequences. What would the numbers on the cards have to be? Would it be possible for the students to lie about these cards? The answer is left as a gift to you, dear reader, to figure out.

- **Some numbers are black and some are red.** These represent 0 and 1 respectively. But what if we could use three colors? Could we then use base-three representation by having Black, Red, and Yellow represent 0, 1, and 2 of each place value? How would we have to construct the cards in such a case? Which numbers could we then represent? Again, the answers to these questions are a gift to the reader and beyond the scope of the Liar's Bingo script offered here.
- **The smallest digit is 0 and the largest is 7. There are no 8's or 9's on any card.** As mentioned above, three places can only represent 0-7 in binary. Representing 8 or 9 in binary (1000 and 1001) requires a fourth place.
- **The largest number is 77.** Per the line above, if the maximum tens or ones digit is 7, then the maximum number must be 77.
- **Black numbers decrease left-to-right and red numbers increase left-to-right.** Reading left to right, suppose one of the cells has a black number in it. In the example in the student notes and reprinted below, the first black number is 63. It is 63 because lying about the cell means saying RRBBRR which corresponds to $40 + 20 + 2 + 1 = 60 + 3$. Moving to the right means decreasing in place-value from 40, to 20, to our next black number (in the case of this strip it is in the 10 place-value or third square from the left) that is 33. Lying about *this* cell means changing the 10 place-value from black to red, adding only 10 to the eventual sum rather than the 40 that we added by changing the first cell to red. We are only turning on the 10s place as opposed to the 40s place before. Hence our eventual magic number is worth $40 - 10 = 30$ less than when we lied about the first cell. So moving left to right, changing black to red means adding smaller and smaller amounts to the sum indicating the "magic number" so it makes sense that the black numbers decrease left-to-right. The case for reds increasing left-to-right is similar.

63	3	33	27	21	22

- **The first three cells (left-to-right) have equal ones digits and the last three cells have the same tens digit (in some cases an implied zero for the tens digit).** Because the first three cells represent the tens place only, the ones digit must be the same. Similar reasoning justifies why the tens digit of the last three cells are equal.